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AN INVESTIGATION OF PRESSURE TRANSIENTS IN PIPELINES WITH TWO-PHASE BUBBLY FLOW

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SUMMARY

This paper presents a two-dimensional model for the analysis of the pressure transient of a two-phase homogeneous bubbly mixture flowing in a pipeline and the numerical integration using the centre implicit method (CIM). Experiments were conducted to confirm the proposed sonic speed equation of an air–water mixture for an air concentration of less than 1%. The 2D CIM model is compared with the method of characteristics (MoC) for a two-phase bubbly flow in a pipeline. The comparisons show that the proposed 2D CIM model generally gives good agreement with the method of characteristics.

KEY WORDS Centre implicit method Pressure transients Two-phase pipe flow Bubbly pipe flow

INTRODUCTION

The investigation of pressure transients in a pipeline for a two-phase flow is more complex than for a single-phase fluid because the presence of the two-phase mixture causes a variation of the local acoustic wave speed in the pipeline. Dijkman and Vreugdenhil[1] examined the pressure transient problem via the gas bubbles cavitation which formed a thin layer at the top of a horizontal pipe, i.e. a separated flow, and found that there were two local acoustic waves propagating to and fro in the pipeline. Swaffield,[2] Kranenburg[3,4] and Driels[5] studied the influence of air–water mixtures in a pipeline during a pressure transient, but they assumed that the two-phase flow phenomenon was concentrated in a region near the valve. Within this region the wave speeds varied, but the wave speed beyond it was constant because of the single-phase liquid assumption. Tullis *et al.*,[6] Wiggert and Sundquist[7] and Wylie[8,9] assumed that a small percentage of free gas was present initially in the pipeline, which remained constant throughout the pressure transient, and that the wave speed was affected by the ratio of air–water mixture and the local pressure. The results of some of the above mentioned researchers show that the pressure transient in two-phase fluids can be non-linear and that the wave propagation velocity is a function of the pressure.

When considering pressure transients involving a two-phase mixture, the speed of propagation of the pressure disturbance is generally lower than that in either the gas or the liquid. Wylie[8] presented a dimensionless graph of wave speed as a function of the absolute pressure ratios for a range of up to 2% by volume of air–water mixtures in a pipeline, for which the ratio of the bulk modulus of the fluid to the pipe material is equal to 0·3. A similar investigation was also conducted by Martin *et al.*[10] for two-phase bubbly and slug flows, but their experiments were conducted with higher values of the void fraction.

Frequently, the pressure transient in two-phase fluids is modelled on the basis of the unsteady one-dimensional conservation equations and solved using the method of characteristics (MoC).

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Discrepancies between the experimental and theoretical results of pressure transients involving such a flow are observed in many investigations. This is due partly to the inadequacy of the governing equations used. However, Streeter[11] reported that the implicit numerical schemes usually perform much faster than the method of characteristics (MoC) in the study of pressure transients in pipelines. This is normally true because there is no restriction on the time step. However, for the given condition the implicit method may not yield satisfactory results, particularly when the transients are sudden and rapid as reported recently by Tan *et al.*[12] and Nathan *et al.*[13]

In this paper a two-dimensional model is presented for describing the pressure transients in a pipeline with air–water bubbly mixtures, which includes the effect of pipe friction. A centre implicit numerical integration scheme[14] is used and the stability criterion of this scheme has been found to satisfy the Courant *et al.*[15] condition. An experiment is also conducted to verify the local acoustic speed equation in the presence of a two-phase air–water bubbly mixture.

BASIC EQUATIONS

For the hydraulic system under consideration, the air–water mixture flowing in the pipe section is assumed to be in thermal equilibrium during the period of transient. Therefore there is no bubble growth due to the latent heat flow at the gas–liquid interfaces. Owing to the low value of the void fraction in pipes, the amount of slip between the gas and liquid phases is negligible. However, the bubbles present in the test section may expand or contract when they are subjected to the expansion or compression waves travelling to and fro in the pipe section. The following assumptions are made with regard to the flow regimes during the transient.

1. Free air and water are present in the pipe system prior to the start of the transient and the amount of air remains unchanged.
2. The air bubbles are uniformly distributed in the pipe as a homogeneous bubbly liquid mixture and the velocities of bubbles and water are assumed to be the same, i.e. no slip.
3. The amount of free air is small (less than 10%), hence the density of the two-phase mixture is dominated by the liquid phase:

$$\rho_m = \alpha\rho_g + (1-\alpha)\rho_f.$$

4. The difference in pressure across a bubble surface is neglected:

$$P_f = P_g.$$

5. The temperature of the fluids remains constant, i.e. an isothermal condition exists.

Using these assumptions, the short-time averages of the continuity and momentum equations for a fluid in a control volume of an air–water mixture flowing in a horizontal pipe are as follows:

continuity

$$\rho_m C^2\left(\frac{\bar{v}}{r}+\frac{\partial\bar{v}}{\partial r}+\frac{\partial\bar{u}}{\partial x}\right)+\frac{\partial\bar{P}}{\partial t}=0, \tag{1}$$

momentum

$$\frac{\partial\bar{u}}{\partial t}+\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{u}}{\partial r}+\frac{\partial\overline{u'2}}{\partial x}+\frac{1}{\rho_m C^2}\overline{u'\frac{\partial P'}{\partial t}}$$

$$=-\frac{1}{\rho_m}\frac{\partial\bar{P}}{\partial x}-g\frac{\partial z}{\partial x}-\frac{1}{r}\frac{\partial(r\overline{u'v'})}{\partial r}+\nu\left(\frac{\partial^2\bar{u}}{\partial x^2}+\frac{\partial^2\bar{u}}{\partial r^2}+\frac{1}{r}\frac{\partial\bar{u}}{\partial r}\right)-\frac{\nu}{3\rho_m C^2}\frac{\partial^2\bar{P}}{\partial x\partial t}, \tag{2}$$

where r is the radial co-ordinate, x the longitudinal co-ordinate, t the time co-ordinate, $\bar{u}$ the axial velocity at r, $\bar{v}$ the radial velocity at r, ρ_m the weighted density, C the acoustic velocity, P the pressure, ν the kinematic viscosity and z the elevation. The prime represents the fluctuations component and the overbar denotes the short-time average component. Assuming the velocity component in the radial direction is much smaller than in the axial direction, i.e. $v \ll u$, the equations can be simplified as follows:

continuity

$$\rho_m C^2\left(\frac{\partial \bar{u}}{\partial x}\right)+\frac{\partial \bar{P}}{\partial t}=0, \tag{3}$$

momentum

$$\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}=-\frac{1}{\rho_m}\frac{\partial \bar{P}}{\partial x}-g\frac{\partial z}{\partial x}+\nu\left(\frac{\partial^2 \bar{u}}{\partial r^2}+\frac{1}{r}\frac{\partial \bar{u}}{\partial r}\right)-\frac{1}{r}\frac{\partial}{\partial r}(r\overline{u'v'}), \tag{4}$$

where $\overline{u'v'}=-\varepsilon\,\partial\bar{u}/\partial r$ is the Reynolds stress term and ε is the eddy viscosity.

Substitution of the Reynolds stress term into the momentum equation yields the following equation:

$$\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}=-\frac{1}{\rho_m}\frac{\mathrm{d}\bar{P}}{\partial t}-g\frac{\partial z}{\partial x}+\left(\frac{\partial \nu'}{\partial r}+\frac{\nu'}{r}\right)\frac{\partial \bar{u}}{\partial r}+\nu'\frac{\partial^2 \bar{u}}{\partial r^2}, \tag{5}$$

where $\nu'=\nu+\varepsilon$ is the effective viscosity. The accurate modelling of the effective viscosity is necessary and will be described in a later section.

Integrating equations (3) and (5) over the cross-sectional area A of the pipe, and using definitions and identities

$$\frac{1}{A}\int_A \bar{P}\,\mathrm{d}A=P, \qquad \frac{1}{A}\int_A \bar{u}\,\mathrm{d}A=U,$$

$$\frac{1}{A}\int_A -\frac{1}{\rho_m}\frac{\mathrm{d}P}{\mathrm{d}x}\mathrm{d}A=-\frac{1}{\rho_m}\frac{\partial P}{\partial x}, \qquad \frac{1}{A}\int_A \bar{u}\frac{\partial \bar{u}}{\partial x}\mathrm{d}A=U\frac{\partial U}{\partial x},$$

$$\frac{1}{A}\int_A\left(\frac{\partial \nu'}{\partial r}+\frac{\nu'}{r}\right)\frac{\partial \bar{u}}{\partial r}\mathrm{d}A=\frac{2}{R^2}\int\frac{\partial \nu'}{\partial r}\frac{\partial \bar{u}}{\partial r}r\mathrm{d}r+\frac{2}{R^2}\int \nu'\frac{\partial \bar{u}}{\partial r}\mathrm{d}r,$$

$$\frac{1}{A}\int \nu'\frac{\partial^2 \bar{u}}{\partial r^2}\mathrm{d}A=\frac{2}{R^2}\left[r\nu'\frac{\partial \bar{u}}{\partial r}\right]_0^R-\frac{2}{R^2}\int \nu'\frac{\mathrm{d}\bar{u}}{\mathrm{d}r}\mathrm{d}r-\frac{2}{R^2}\int r\frac{\partial \bar{u}}{\partial r}\frac{\partial \nu'}{\partial r}\mathrm{d}r,$$

the continuity and momentum equations can be reduced to the following forms:

$$\frac{1}{\rho_m C^2}\frac{\partial P}{\partial t}+\frac{\partial U}{\partial x}=0, \tag{6}$$

$$\frac{\partial U}{\partial x}+U\frac{\partial U}{\partial x}+\frac{1}{\rho_m}\frac{\partial P}{\partial x}+\frac{2}{R\rho_m}\tau=0, \tag{7}$$

where R is the radius of the pipe and τ is the wall shear stress. The gravity term is neglected in equation (7) because the pipe is a horizontal one. The shear stress $\tau=-\rho\nu'(\partial U/\partial r)_R$ follows a four-

Table I. Definition of the four-region model

Region	Y^*	k	B
I	0–5	0	1
II	5–30	0·2	0
III	30–0·175R^*	0·4	0
IV	0·175–R^*	0	0·07R^*

$u^* = \sqrt{(\tau/\rho)}$, $\tau = -\rho\nu'(\partial u/\partial r)_R$, $Y = R - r$, $Y^* = u^* r/\nu$, $R^* = u^* R/\nu$.

region model described by Ohmi and Usui[16] and the distribution of the effective viscosity ν' is defined by

$$\nu' = \nu[kY^* + B]. \tag{8}$$

The values of k, B and Y^* of the four-region model are given in Table I.

Local sonic speed

In a two-phase flow situation the governing equations (6) and (7) are incomplete if the constitutive equation for describing the local sonic speed is not defined. For a single-phase flow the local sonic speed C_0 is well known and is given as[8,11]

$$\frac{1}{C_0^2} = \rho\left(\frac{1}{K} + \frac{D}{TE}\right), \tag{9}$$

where K is the bulk modulus of the liquid, D the pipe diameter, T the pipe thickness and E the Young's modulus of the pipe material. However, the bulk modulus of the homogeneous bubbly mixture can be expressed as follows:

$$\frac{1}{K} = \frac{1}{K_f}(1-\alpha) + \alpha\left(\frac{1}{K_g}\right), \tag{10}$$

where α is the void fraction and K_f and K_g are the bulk modulus of the liquid and vapour phases respectively. Using the ideal-gas law, it can be shown that the bulk modulus of the vapour phase, $K_g \simeq 1/P$. Since K_f is larger than the local pressure P, equation (10) can be simplified to

$$\frac{1}{K} = \frac{1}{K_f} + \frac{\alpha}{P}. \tag{11}$$

Substituting equation (11) into equation (9) gives the expression for the sonic velocity of a homogeneous mixture:

$$\frac{1}{C^2} = \rho_m\left(\frac{1}{K_f} + \frac{\alpha}{P} + \frac{D}{KE}\right). \tag{12}$$

For a transient situation the change in the void fraction is assumed to be an inverse function of the local pressure:

$$\alpha = \alpha_0\frac{P_0}{P}, \tag{13}$$

where P_0 and α_0 are the initial pressure and void fraction respectively. Hence equation (12) becomes

$$\frac{1}{C^2} = \rho_m\left(\frac{1}{K_f} + \alpha_0\frac{P_0}{P^2} + \frac{D}{KE}\right). \tag{14}$$

From the above equation, the local sonic speeds for two-phase flow depend on the local pressure as well as on those terms given by equation (9). The validity of the proposed equation is verified later by experimentation.

FINITE DIFFERENCE SCHEME

Equations (6), (7), (8) and (14) are solved simultaneously using the centre implicit method, which is now extended for application in the two-phase bubbly flow regime in a horizontal pipe. If the pipe section is divided into $N-1$ lengths with fixed axial interval Δx, the time increment on a space–time domain at a local node i is therefore determined by $\Delta t_i = \Delta x_i / C_i$, where C_i is the local wave velocity. The finite difference quotients of the practical derivatives, expressed in terms of a variable ϕ, are given as follows:

time domain

$$\left(\frac{\partial \phi}{\partial t}\right)_r = \frac{1}{2}\left(\frac{\phi_i^{t+\Delta t_i} - \phi_i^t}{\Delta t_i} + \frac{\phi_{i+1}^{t+\Delta t_{i+1}} - \phi_{i+1}^t}{\Delta t_{i+1}}\right), \tag{15}$$

axial domain

$$\left(\frac{\partial \phi}{\partial x}\right)_r = \frac{(2-\theta)(\phi_{i+1}^t - \phi_i^t) + \theta(\phi_{i+1}^{t+\Delta t_{i+1}} - \phi_i^{t+\Delta t_i})}{2\Delta x}, \tag{16}$$

where ϕ is the variable P or U, θ is the artificial viscosity and r is the radial distance from the centre of the pipe. Note that the parameter θ is introduced directly to the spatial derivative terms only because the mathematical solution of the equations, in the given forms, can be easily handled.[14] Equations (15) and (16) are valid for a particular radial distance r from the centreline of the pipe section. The variations of the velocity U in the radial direction are given as follows:

$$\frac{\partial U}{\partial r} = \frac{U_{i,r+1}^{t+\Delta t} - U_{i,r-1}^{t+\Delta t}}{2(\Delta r)}, \tag{17}$$

$$\frac{\partial^2 U}{\partial r^2} = \frac{U_{i,r+1}^{t+\Delta t} + U_{i,r-1}^{t+\Delta t} + 2U_{i,r}^{t+\Delta t}}{(\Delta r)^2}, \tag{18}$$

$$U = (U_{i,r}^t + U_{i+1,r}^t)/2. \tag{19}$$

We substitute equations (15)–(19) into the momentum equation to solve for the values of U and P. Details of the 2D solver have been described in detail by Nathan *et al.*[13] in an earlier publication.

From these equations it can be seen that the values of U and P computed for the nodes in the axial domain may not be synchronized in the time domain because of the slight difference in the local wave velocities at these nodes. Therefore an interpolation scheme for the time domain is necessary. These interpolation routines for the pressure P and the mean velocity U at each node i and time instant $t+\Delta t_i$ are given as follows:

$$\phi_i^{t+\Delta t_{\min}} = \phi_i^t + \left(\frac{\phi_i^{t+\Delta t_i} - \phi_i^t}{\Delta t_i}\right)\Delta t_{\min}, \tag{20}$$

where $\phi = P$ or U.

The initial steady state values of the pressure and velocity in the pipeline are used as the initial boundary conditions. With the initial boundary conditions known, the local velocity profile can also be computed and hence the value of the wall shear stress τ. The local wave speed C_i for the axial length of the pipeline is also calculated and from these values of C_i the time step increment ΔT_i can be determined. The values of the local pressure and velocity at the new time step are then calculated using the governing equations. The approach adopted in this scheme is implicit in nature because the values of P and U are obtained by solving the equations simultaneously.

Because of the different values of local sonic speed at the nodes, a minimum time step ΔT_{min} is determined and the new values of P and U at the new time step are obtained by using the interpolation equation (20). The computational procedure is then repeated by computing the new values of the wall shear stress τ and the velocity profile. The procedure for the prediction of the pressure transient of a two-phase mixture in a pipeline is summarized in the flowchart shown in Figure 1.

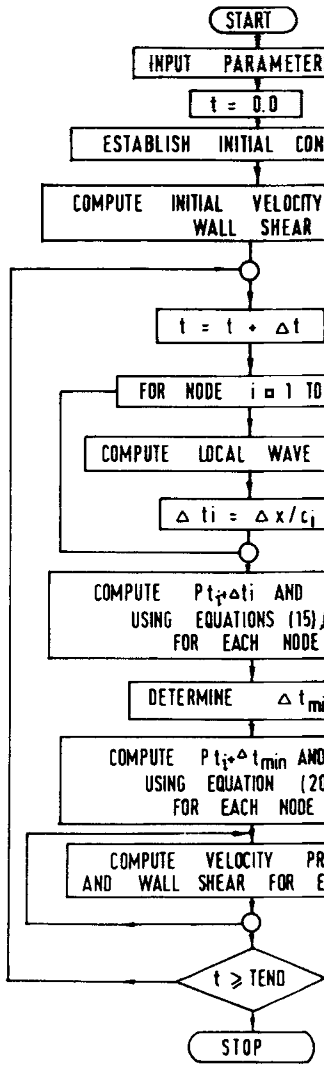

Figure 1. Procedure for the prediction of pressure transients

EXPERIMENTAL INVESTIGATION

For the purpose of verification of the expression for the local sonic speeds in a two-phase bubbly mixture, an experiment was carried out in a vertical pipe test section 2 m long and 47·5 mm in diameter. A series of tests with air bubbles–water mixtures was conducted. Figure 2 shows a schematic diagram of the test rig. A vertical test section was used because of the ease of generating the two-phase bubbly mixture. Since the transient is of the order of milliseconds, the effect of bubbles coalescing and drifting to the top of the test section is negligible. Two sensitive piezoresistive pressure transducers with resolution 0·5 kN mm^{-2} were placed at the top and bottom of the test section.

The pressure in the reservoir was maintained constant by supplying compressed air to it. The two-phase bubbly flow regimes in the test section were maintained by injecting air at the bottom of the test section through an injector assembly. The pressure transient in the section was affected by a sudden closure of the valves, and the pressure responses at the transducers were recorded by a microcomputer data acquisition system. The high-speed data acquisition system is needed for data collection and analysis. The initial steady state flow rate was estimated by timing the fall of the water level in the reservoir after gate valve no. 3 was closed. It should be emphasized that the fall of the water level has a negligible effect on the system pressure due to the high initial pressure of the compressed air within the system. At the end of the test, the two-phase void fraction was determined simply by recording the volume of air trapped in the top of the test section.

RESULTS AND DISCUSSION

Table II shows the results of five experimental runs conducted with two-phase bubbly flow conditions in the test section prior to the onset of pressure transients. The bubbles in the flow are homogeneously distributed and generally less than 1% by volume, i.e. $\alpha < 0{\cdot}01$. The wave speed C is determined experimentally by reading off the time difference between the two pressure traces of the transducers since the length of the test section has been fixed at 2 m. The pressure response of a typical test is shown in Figure 3. It can be seen that the pressure fluctuations for two-phase flow

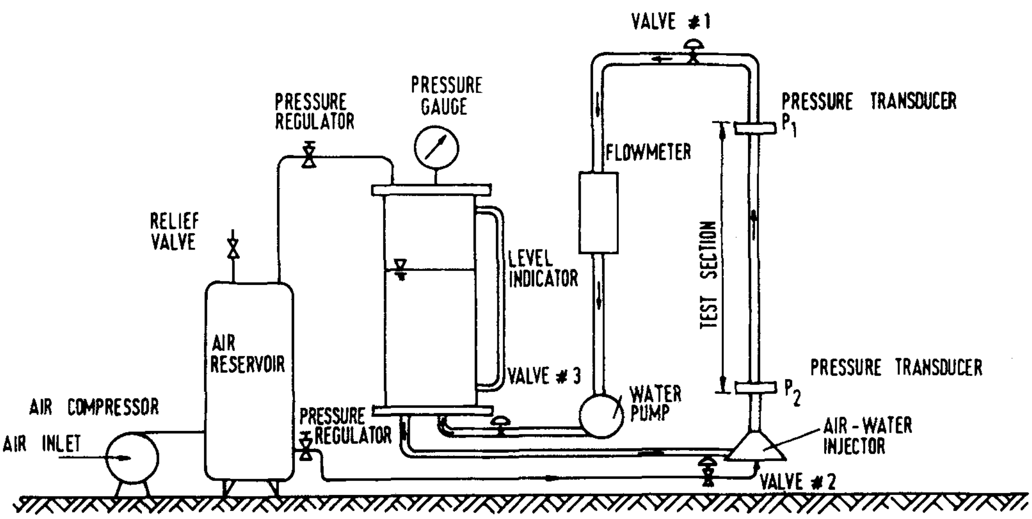

Figure 2. Schematic diagram of the experimental test facility

Table II. Comparison of theoretical wave speed with experimental data

Test run no.	Pressure of air column (bar)	Void fraction α (%)	Δt (ms)	Wave velocity: Experimental C_E (m s^{-1})	Wave velocity: Theory (equation (7)) C_T (m s^{-1})	C_T/C_E
1	2·373	0·8	14·39	139	141·7	1·019
2	1·633	0·73	15·83	126·3	123	0·974
3	2·149	0·85	14·73	136	130	0·956
4	2·433	0·65	12·58	159	160·5	1·009
5	1·88	0·68	14·29	140	138·0	0·985

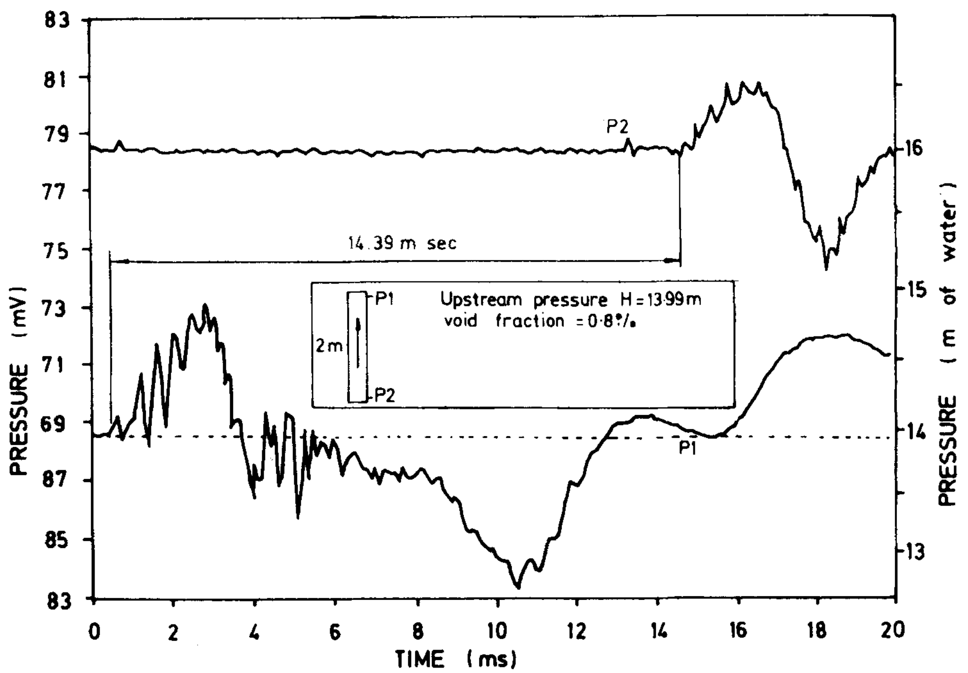

Figure 3. Typical pressure response in the test section

are generally larger than for single-phase liquid in a system.[14] It was observed that the air bubbles deformed from their spherical shape during the transient period. After the transient, the bubbles moved slowly to the top of the test section due to the effect of buoyancy, which is shown by the deviation of the pressure reading P_1 from its original value. However, the change in the pressure readings will not affect the accuracy of the local sonic speed because it takes only a few milliseconds for the wave to move across the test section.

The comparison of the theoretically predicted local sonic speed with that obtained from the experiment shows good agreement, i.e. the maximum error in the prediction of C is 4·5%. This shows that the constitutive equation, i.e. equation (8), used to predict the local sonic speed is valid for the range of tests conducted. It should be emphasized that accurate prediction of the local sonic speed is important for the investigation of pressure transients using the proposed model.

Further verification of the two-dimensional two-phase flow CIM model by comparison with the predictions of Wylie[9] and Padmanabhan *et al.*[17] was carried out. In their analysis they used a time-line interpolation scheme together with the method of characteristics for the prediction of the experimental results. However, the governing equations used in their model were basically one-dimensional. The mass of air in the pipe was present before the start of the transient and was uniformly distributed throughout during the pressure transient. Table III gives the details of the two numerical experiments performed by them. In the proposed CIM model the conservations equations employed are two-dimensional and the local sonic speeds vary with the value of the void fraction. The number of nodes used in the simulation is 31 and the value of the artificial viscosity θ used is 1·005.[12]

Figures 4 and 5 show the comparison of the predicted results between the 2D two-phase flow CIM model and the time-line method used by Wylie. For an initial value of $\alpha = 0{\cdot}002$ the pressure head comparison at a distant to length ratio x/L of 0·4 shows a better pressure head time history: there is no overshoot at $t = 12$ s and the response of this trace is more sensitive at two time instances, namely 2 s and 8 s after the start of the transient. Similarly, the comparison given by another test run, with $\alpha = 0{\cdot}009$, shows good agreement in the pressure head time at an axial location ratio of $x/L = 0{\cdot}733$.

In a similar comparison with the numerical calculations carried out by Padmanabhan *et al.*,[17] the two-phase CIM model gives better predictions as shown by Figures 6 and 7. For an initial value of $\alpha = 0{\cdot}014$ the pressure ratio–time history at the axial locations $x/L = 0{\cdot}75$ and 0·25 shows good agreement during the pressure transients. The CIM model again appears to be more sensitive, as indicated by the presence of some ripples at the start of the transient in the time

Table III. Details of the pipe systems studied by Wylie and Padmanabhan *et al.*

Data of Wylie[9]	Case 1	Case 2
Diameter (m)	0·61	0·61
Length (m)	3000	3000
Sonic speed (m s^{-1})	981·4	981·4
Initial pressure (kg m^{-2})	992	992
Initial velocity (m s^{-1})	0·89	0·89
Temperature (K)	288	288
Density (kg m^{-3})	992	992
Downstream pressure head (m)	60	13·4
Upstream pressure head fluctuations (m)	60–10·3 in 0·204 s	60·75–10·4 in 0·204 s
Friction coefficient	0·0	0·005
Void fraction α (%)	0·2	0·9
Data of Padmanabhan *et al.*[17]		
Diameter (m)	0·025	
Length (m)	18·0	
Sonic speed in water only (m s^{-1})	600	
Initial velocity (m s^{-1})	1·5	
Void fraction α (%)	0·0145	
Reservoir pressure (MPa)	0·38	
Axial step (m)	0·6	
Radial step (m)	$6{\cdot}58 \times 10^{-4}$	
Friction coefficient	0·00099	

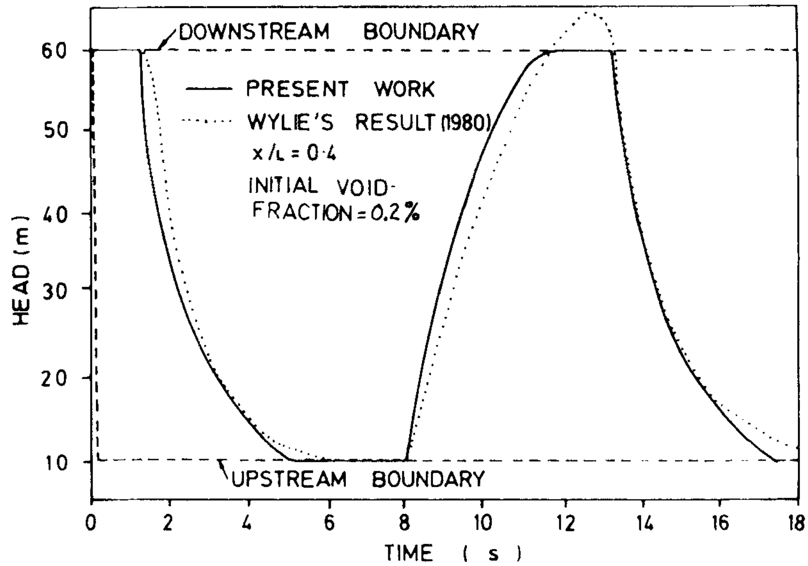

Figure 4. Comparison of the pressure head time history at $x/L = 0{\cdot}4$

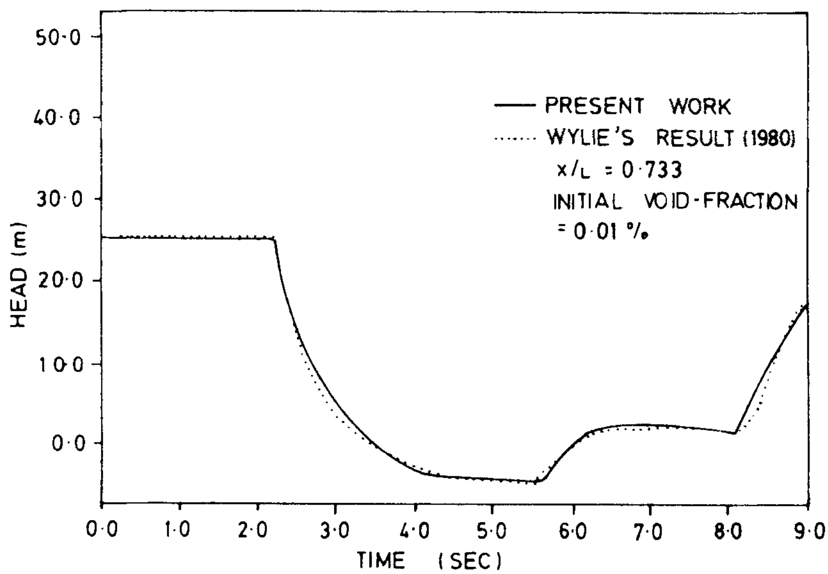

Figure 5. Comparison of the pressure head time history at $x/L = 0{\cdot}733$

interval of less than 0·1 s and by less truncation of the subsequent pressure peaks. These comparisons show that the 2D two-phase flow CIM model is a better numerical routine because it gives a more accurate assessment of the flow conditions in the pipe during transients.

CONCLUSIONS

From the work carried out in this investigation, it is found that the acoustic wave velocities of bubbly two-phase mixtures in a pipe can be accurately predicted by a simple equation with the

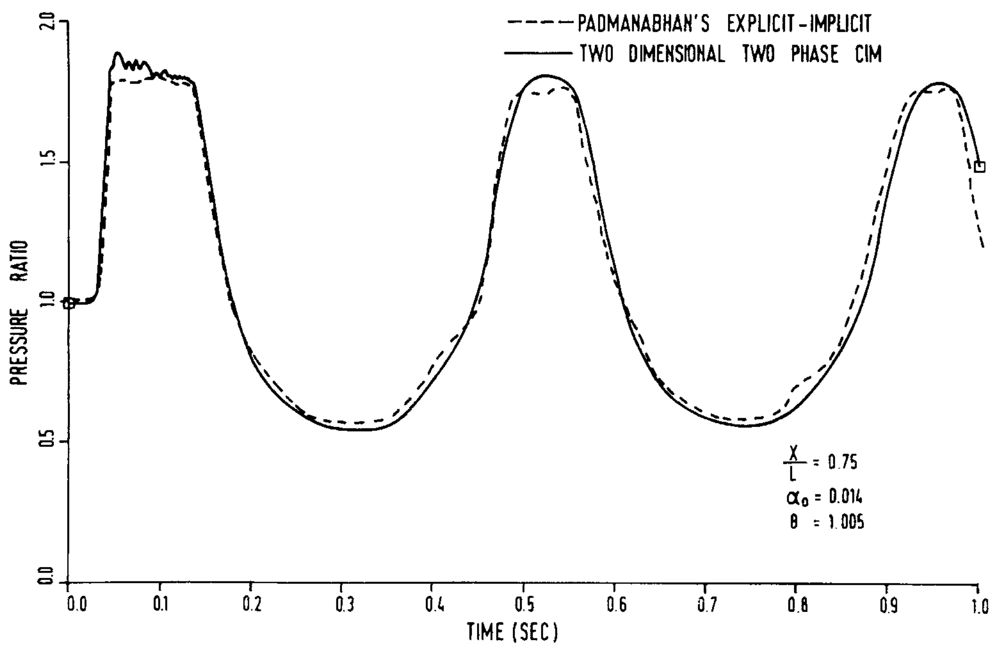

Figure 6. Pressure ratio versus time curve

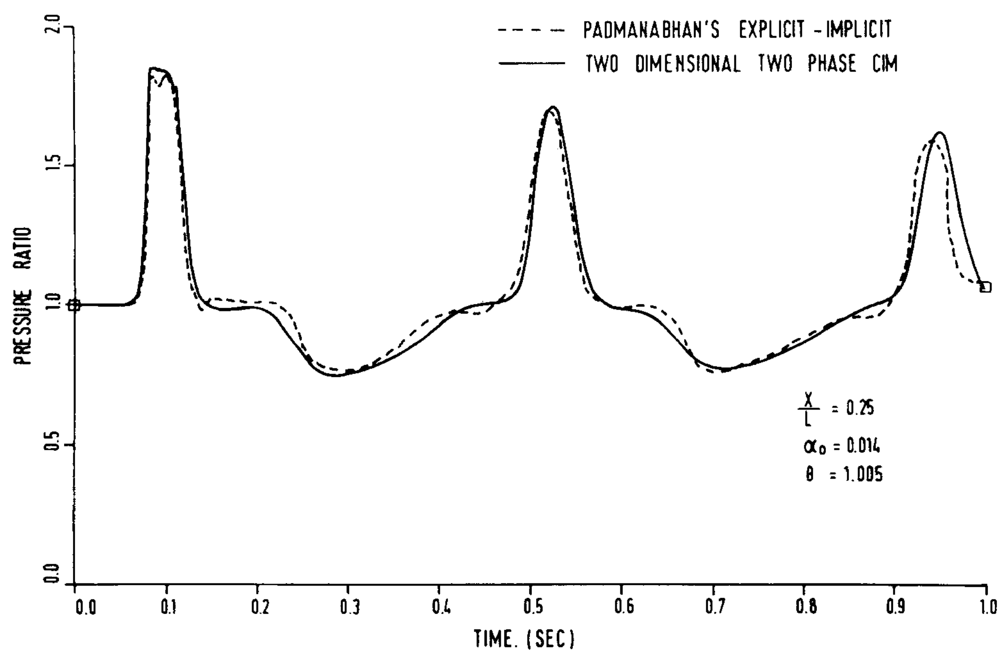

Figure 7. Pressure ratio versus time curve

initial void fraction as a parameter. The comparison of the sonic speeds with the experiments gives good agreement, mainly because the void fraction in the pipeline is small, i.e. less than 1%. The accurate prediction of the local sonic speed is necessary for two-phase flow because it will affect the accuracy of the proposed theoretical model for the prediction of pressure transients in the pipeline.

The comparison of the predicted results with those of Wylie[9] and Padmanabhan *et al.*[17] shows excellent agreement overall, and perhaps in many ways the CIM model shows more sensitivity during the pressure transients. This is because the governing equations used by the proposed model are two-dimensional and the model also includes both the effect of the velocity profile in the flow of the two-phase bubbly flow mixture and the effect of the varying local sonic speed. However, it should be pointed out that although the comparisons agree well with each other, they are all results from numerical methods and not from actual experimental data of pressure transients. Therefore the real challenge now is for the authors to apply the CIM model to experimental data that are available in the literature.

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APPENDIX: NOTATION

A	cross-sectional area of pipe
B	intercept of pipe friction model equation
C	sonic velocity
D	diameter of pipe
E	Young's modulus of the pipe material
H	pressure head in height of water column
K	bulk modulus of liquid
k	gradient of pipe friction model equation
N	number of nodes in axial direction
P	pressure
$\bar{P}$	short-time average of P
R	radius of pipe
r	radial co-ordinate
T	thickness of pipe
t	time
U	mean axial velocity
u	axial velocity at radius r
$\bar{u}$	short-time average of u
V	volume
v	radial velocity at radius r
$\bar{v}$	short-time average of v
x	spatial length co-ordinate
Y	radial length difference, $R-r$
Y^*	non-dimensional length
z	elevation
α	void fraction

ε	eddy viscosity
μ	viscosity
ν	kinematic viscosity
ν'	total viscosity, $\varepsilon + \nu$
θ	artificial viscosity
ρ	density of the liquid
ϕ	variable representing either P or U
τ	wall shear stress

Subscripts

f	liquid phase
g	vapour phase
m	mean weighted value

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